Midsemestral exam 2010
M.Math.IInd year
Advanced Number Theory — B.Sury
Open Book: Borevich-Shafarevich

Q 1. Let $f(x_1, \cdots, x_n) = \sum_{i=1}^{n} a_i x_i^2$ where $a_i \in \mathbf{F}_p^*$ with $p \neq 2$. Determine the number of solutions $(x_1, \cdots, x_n) \in \mathbf{F}_p^n$ of $f = 0$.

Q 2. Let p be an odd prime and $(a_i, b_i) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}; 1 \leq i \leq 3p-2$. Use the Chevalley-Warning theorem to obtain a set T of cardinality p or $2p$ for which $\sum_{i \in T}(a_i, b_i) = (0,0) \in \mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p\mathbf{Z}$.

Q 3.
(i) If $p > 3$, prove that $\sum_{(\frac{a}{p})=1} a \equiv 0 \bmod p$.
(ii) Use the quadratic reciprocity law to decide whether $x^2 + 5x \equiv 12 \bmod 31$ has solutions.

Q 4. Consider the Hilbert symbol (a,b) on $\mathbf{Q}_p$ where p is an odd prime. Write $(a,b) = (-1)^{[a,b]}$ where $[a,b] \in \mathbf{Z}/2\mathbf{Z}$. Determine the matrix $\begin{pmatrix} [p,p] & [p,u] \\ [u,u] & [u,u] \end{pmatrix}$ where u is a p-adic unit whose image in $\mathbf{F}_p$ is a quadratic non-residue.

Q 5. Prove that the polynomial $(X^2-2)(X^2-17)(X^2-34)$ has a root in each $\mathbf{Q}_p$.
Hint: You may use Hensel's lemma.

OR

Use Hensel's lemma to prove that $\alpha \in \mathbf{Q}_p$ is a p-adic unit if and only if α has an n-th root for each n relatively prime to $p(p-1)$.

Q 6. Let $f = a_0 + a_1 X + \cdots + a_n X^n \in \mathbf{Q}_p[X]$ be irreducible of degree $n \geq 1$, where $f(0) \neq 0$. Use Hensel's lemma to prove that if $a_0, a_n \in \mathbf{Z}_p$, then $f \in \mathbf{Z}_p[X]$.

OR

Let $f = \sum_{i=1}^{4} a_i x_i^2$ with $a_i \in \mathbf{Q}_p^*$. If f does not represent 0, prove that $\prod_{i=1}^{4} a_i$ is a square and that Hasse invariant $c(f) = -1$.

Q 7. Let a be an integer and p be a prime not dividing a. Prove that the sequence $\{a^{p^n}\}$ converges in $\mathbf{Q}_p$ to a $(p-1)$-th root of unity.

OR

If $\sum_{n=1}^{\infty} a_n$ is a convergent series in $\mathbf{Q}_p$, then any rearrangement also converges to the same sum.

Q 8. Prove that $\mathbf{Q}_p$ and $\mathbf{Q}_q$ are non-isomorphic fields when $p \neq q$.